B.B. 6.7 (Electromagnetic waves in the ionosphere)

The ionosphere acts as a dielectric medium with refractive index $n = \left[1 - (\omega_p/\omega)^2\right]^{1/2}$ where ω_p is the plasma frequency. What is the dispersion relation $k(\omega)$? And what are the phase and group velocities, $v_{phase} = \frac{\omega}{k}$ and $v_g = \frac{\partial \omega}{\partial k}$?

Recall that $k^2 = \omega^2 \mu_0 \epsilon_0 K_e \cancel{K_m}^{1} - j\omega\mu_0 K_m \sigma$ (6.42)

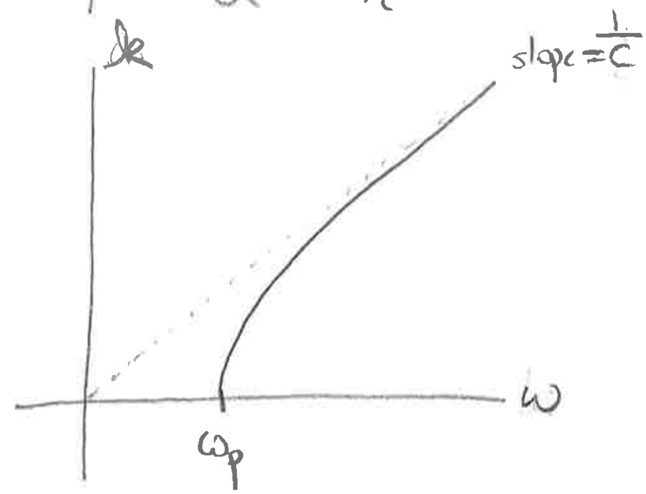
This implies that k has real and imaginary components (6.42b). In this problem, we take the approach of treating the plasma as having a (potentially) complex refractive index. That is: we treat it as a dielectric with some degree of absorption.

According to this picture, $v_{phase} = \frac{\omega}{k} = \frac{c}{n}$

$$v_{phase} = \frac{c}{\sqrt{1-(\omega_p/\omega)^2}} = \frac{\omega}{k}$$

$$k = \frac{\omega}{c}\sqrt{1-(\omega_p/\omega)^2}$$

If $\omega < \omega_p$ then k = imaginary (absorption)



What is the group velocity?

$$v_g \equiv \frac{\partial \omega}{\partial k} = \left(\frac{\partial k}{\partial \omega}\right)^{-1}$$

$$\frac{\partial k}{\partial \omega} = \frac{1}{c}\sqrt{1-\left(\omega_p/\omega\right)^2} + \frac{\omega}{c}\cdot\frac{\frac{1}{2}\left(+2\frac{\omega_p}{\omega}\right)\left(+\frac{\omega_p}{\omega^2}\right)}{\sqrt{1-\left(\omega_p/\omega\right)^2}}$$

$$= \frac{1}{c}\left\{\sqrt{1-\left(\omega_p/\omega\right)^2} + \left(\frac{\omega_p}{\omega}\right)^2\frac{1}{\sqrt{1-\left(\omega_p/\omega\right)^2}}\right\}$$

$$= \frac{1}{c}\;\frac{1}{\sqrt{1-\left(\omega_p/\omega\right)^2}}\left\{\left(1-\cancel{\left(\frac{\omega_p}{\omega}\right)^2}\right) + \cancel{\left(\frac{\omega_p}{\omega}\right)^2}\right\}$$

$$\frac{\partial \omega}{\partial k} = \boxed{\frac{1}{c}\;\frac{1}{\sqrt{1-\left(\omega_p/\omega\right)^2}} = \frac{1}{v_g}}$$

$$\boxed{\begin{aligned} v_{group} &= c\sqrt{1-\left(\omega_p/\omega\right)^2} \\ v_{phase} &= c\;\frac{1}{\sqrt{1-\left(\omega_p/\omega\right)^2}} \end{aligned}}$$

If $\omega < \omega_p$ then the phase & group velocities become imaginary (absorption)

If $\boxed{\omega = \sqrt{2}\,\omega_p}$ then $\boxed{\begin{aligned} v_g &= c/\sqrt{2} \\ v_p &= \sqrt{2}\,c \end{aligned}}$