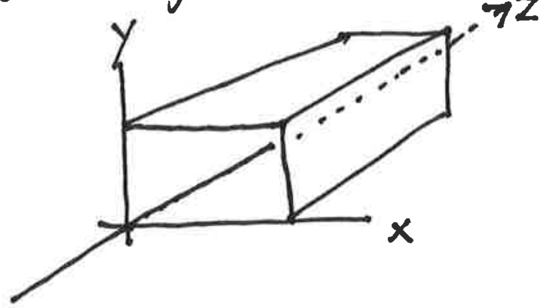
B.B. 5.4 (TEM modes in a rectangular waveguide)

In this problem, we will calculate the minimum frequency for propagation of TEM (transverse electromagnetic modes) in a rectangular waveguide like this:



We are given the y-directed electric field:

$$E_y = E_{oy} \sin(k_x x)\cos(k_y y)\cos(\omega t - k_z z)$$

Since this must obey a wave equation, there is a relationship between k_x, k_y, k_z and ω.

$$\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) E_y = \frac{1}{c^2}\partial_t^2 E_y \quad \leftarrow \text{wave eqn.}$$

$$\left(-k_x^2 - k_y^2 - k_z^2\right) E_y = \frac{1}{c^2}(-\omega^2) E_y \quad \leftarrow \text{take derivatives}$$

So $$\boxed{k_x^2 + k_y^2 + k_z^2 - \frac{\omega^2}{c^2} = 0}$$

- Since the electric field E_y must ~~ϕ~~ vanish at the boundaries $x=0$ & $x=a$ (its a conducting box so $E_y=0$ inside metal and the tangential Electric field is continuous across a boundary) we know that

$$\boxed{k_x a = l\pi} \quad \text{or} \quad \boxed{k_x = \frac{l\pi}{a}} \quad l = 0,1,2\ldots$$

- What about the x-directed Electric field, by the way? We know that in the waveguide $\bar{\nabla}\cdot\bar{E}=0$ (no charge, except possibly on surfaces), so

$$\partial_x E_x + \partial_y E_y + \cancel{\partial_z E_z}^{\,0} = 0 \quad \text{since no } E_z \text{ (its a TE mode)}$$

so $\partial_x E_x = -\partial_y E_y$

$$\partial_y E_y = -E_{oy} k_y \sin(k_x x)\sin(k_y y)\cos(\omega t - k_z z) = -\partial_x E_x$$

Integrate to find E_x...

$$E_x = \int dx\,(-\partial_y E_y) = \frac{-k_y}{k_x} E_{oy}\cos(k_x x)\sin(k_y y)\cos(\omega t - k_z z)$$

But $E_x = 0$ @ $y=0$ and $y=b$ (for same reason that $E_y=0$ @ $x=0$ & $x=a$)

so $k_y b = m\pi \Rightarrow \boxed{k_y = \frac{m\pi}{b}}$

- So:

$$\boxed{\begin{aligned} E_x &= -E_{oy}\left(\frac{m}{l}\right)\left(\frac{a}{b}\right)\cos\left(\frac{\pi l}{a}x\right)\sin\left(\left(\frac{\pi m}{b}\right)y\right)\cos(\omega t - k_z z) \\ E_y &= E_{oy}\sin\left(\frac{\pi l}{a}x\right)\cos\left(\frac{\pi m}{b}y\right)\cos(\omega t - k_z z) \end{aligned}}$$

- I can now use this to find the dispersion relation. Since

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

$$k_z^2 = \frac{\omega^2}{c^2} - \left(\frac{l\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2$$

$$k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{l\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}$$

- Note that if $\left(\frac{\omega}{c}\right)^2 < \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$ then k_z becomes purely imaginary, and there is no propagation of waves down the waveguide. The ~~cutoff~~ cutoff frequency is

$$\boxed{\omega_c = c\sqrt{\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}}$$ cutoff frequency

The phase velocity $v_{phase} = \frac{\omega}{k_z}$ is

$$\boxed{v_{phase} = c\left(\frac{\omega}{\sqrt{\omega^2 - \omega_c^2}}\right)}$$ phase velocity