BB 4.4 (Stability of the atom)

If electrodynamics is correct, and the planetary model of the atom is correct, then the atom would not be stable. This is because, according to the planetary model, orbiting electrons are constantly accelerating. And according to electrodynamics, accelerating charges radiate power according to Larmor's formula. In this problem we will compute the expected lifetime of a hydrogen atom given these two assumptions.

We will need to compute the rate that the orbital radius changes due to radiation.

$$\left(\frac{dr}{dt}\right) = \left(\frac{dr}{dE}\right)\left(\frac{dE}{dt}\right)$$

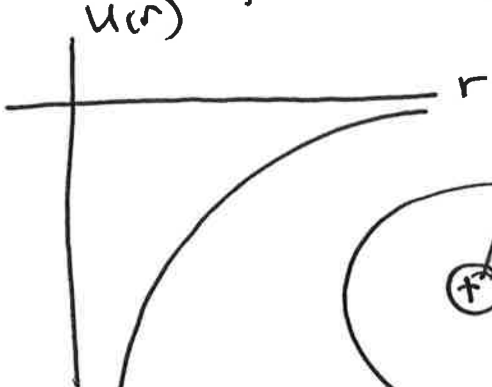
rate of change of orbital radius

how the radius changes as energy changes

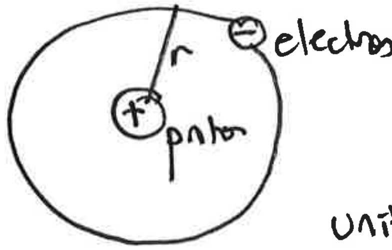
how much energy is radiated away per unit time

First, we will consider how the radius changes with energy. We will need to compute $r(E)$ or, equivalently, $E(r)$.

- The potential energy of an electron at distance r from a proton is given by $U(r)$



$$U(r) = -\frac{q_e q_p}{4\pi\epsilon_0 r}$$



Since both have unit charge e, lets write this as

$$U(r) = \frac{-e^2}{4\pi\epsilon_0 r}$$

- The kinetic energy of the orbiting electron is given by $K = \frac{1}{2} m v^2 = \frac{1}{2} m r^2 \omega^2$ where ω is the angular velocity and m is the electron mass.

- The kinetic energy can be written in terms of r by using Newton's second law to eliminate ω.

(N2L) $F_c = mv^2/r$

$$\frac{e^2}{4\pi\epsilon_0 r^2} = m r \omega^2 \quad \omega^2 = \frac{e^2}{4\pi\epsilon_0 m r^3}$$

Coulomb's law

- Now the total energy of the orbiting electron is

$$E(r) = \frac{-e^2}{4\pi\epsilon_0 r} + \frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r}$$

$$E(r) = -\frac{e^2}{8\pi\epsilon_0 r}$$

- How does E change as r changes?

$$\left(\frac{dE(r)}{dr}\right) = \frac{e^2}{8\pi\epsilon_0 r^2} \quad \text{so} \quad \frac{dr}{dE} = \frac{8\pi\epsilon_0 r^2}{e^2}$$

- What, now, is (dE/dt)? we use Larmor's formula $\frac{dE}{dt} = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}$

The centripetal acceleration is

$$a = \omega^2 r$$

So $$\frac{dE}{dt} = \frac{q^2\omega^4 r^2}{6\pi\epsilon_0 c^3}$$

$$= \frac{q^2 r^2}{6\pi\epsilon_0 c^3}\frac{e^4}{16\pi^2\epsilon_0^2 m^2 r^6}$$

But q is just e, so let's simplify....

$$\frac{dE}{dt} = \frac{-e^6}{96\pi^3 \epsilon_0^3 c^3 m^2 r^4}$$

This is ⊖ since the atom is losing energy.

- Now lets find $\left(\frac{dr}{dt}\right) = \left(\frac{dr}{dE}\right)\left(\frac{dE}{dt}\right)$

$$\left(\frac{dr}{dt}\right) = -\left(\frac{8\pi\epsilon_0 r^2}{e^2}\right)\left(\frac{e^6}{96\pi^3 \epsilon_0^3 c^3 m^2 r^4}\right)$$

$$\left(\frac{dr}{dt}\right) = \frac{-e^4}{12 r^2 \epsilon_0^2 \pi^2 c^3 m^2}$$

- Now we are in a position to separate variables and integrate to find the time, T, it takes to decay from an initial radius, r_0, to zero radius.

$$\int_{r_0}^{0} r^2\,dr = \int_0^T \frac{-e^4\,dt}{12\epsilon_0^2 \pi^2 c^3 m^2}$$

$$-\frac{r_0^3}{3} = \frac{-e^4}{12\epsilon_0^2 \pi^2 c^3 m^2}\,T$$

$$T = \frac{4\epsilon_0^2 \pi^2 c^3 m^2 r_0^3}{e^4}$$

If we use r_0 as the Bohr radius, then

$T \approx \boxed{15 \text{ pico-seconds}}$, or 0.015 nano seconds