Ex 3.2 (Gradient, divergence, curl)

- For each of the following vector fields, calculate the curl & the divergence. If $\bar{\nabla}\times\bar{V}=0$, then obtain a scalar function, Φ, from which $\bar{V}$ can be derived...

(a) $\bar{F} = \underbrace{(x+y)}_{F_x}\hat{x} + \underbrace{(-x+y)}_{F_y}\hat{y} + \underbrace{(-2z)}_{F_z}\hat{z}$

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ (x+y) & (-x+y) & (-2z) \end{vmatrix} = 0(\hat{x}) + 0(\hat{y}) - 2\hat{z} \neq 0$$

$\bar{\nabla}\times\bar{F} = -2\hat{z}$ This means F is not conservative and we cannot define $\Phi(x,y,z)$.

$$\bar{\nabla}\cdot\bar{F} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left((x+y)\hat{x} + (-x+y)\hat{y} + (-2z)\hat{z}\right)$$

$$= 1 + 1 - 2 = 0$$

So $\bar{\nabla}\cdot\bar{F} = 0$ and it is a divergence-free vector field.

(b) $\bar{G} = (2y)\hat{x} + (2x+3z)\hat{y} + (3y)\hat{z}$

$$\nabla \times \bar{G} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 2y & 2x+3z & 3y \end{vmatrix} = \hat{x}(3-3) + \hat{y}(0-0) + \hat{z}(2-2) = 0$$

$\nabla \times \bar{G} = 0$ (so curl-less)

$\nabla \cdot \bar{G} = 0 + 0 + 0 = 0$ (and divergence-less)

- Since $\nabla \times \bar{G} = 0$, lets try to find $\Phi(x,y,z)$ by integrating $\bar{G}$ along the x-axis, y axis & z-axis...

$$\int \bar{G} \cdot d\bar{s} = \int (2y)\,dx + \int (2x+3z)\,dy + \int (3y)\,dz$$

$= 2xy$ (this is how it depends on x)

$+ 2xy + 3yz$ (this is how it depends on y)

$+ 3yz$ (this is how it depends on z)

- So let's guess that $\boxed{\Phi(x,y,z) = -2xy - 3yz}$

And let's check: $G_x = -\frac{\partial\Phi}{\partial x} = 2y$ ✓

$G_y = -\frac{\partial\Phi}{\partial y} = 2x + 3z$ ✓

$G_z = -\frac{\partial\Phi}{\partial z} = 3y$ ✓

(b) $\bar{H} = (x^2 - z^2)\hat{x} + (2)\hat{y} + (2xz)\hat{z}$

$$\bar{\nabla} \times \bar{H} = \hat{x}(\partial_y H_z - \partial_z H_y) + \hat{y}(\partial_z H_x - \partial_x H_z) + \hat{z}(\partial_x H_y - \partial_y H_x)$$

$$= \hat{x}(0 - 0) + \hat{y}(-2z - 2z) + \hat{z}(0 - 0)$$

$$\bar{\nabla} \times \bar{H} = -4z$$

$$\bar{\nabla} \cdot \bar{H} = \partial_x H_x + \partial_y H_y + \partial_z H_z$$

$$= 2x + 0 + 2x = 4x$$

$$\bar{\nabla} \cdot \bar{H} = 4x$$