Ex 3.1 (Electric field, electric potential & line integrals)

- Consider an electric field given by

$$\bar{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$

where

$$E_x = 6xy$$
$$E_y = 3x^2 - 3y^2$$
$$E_z = 0$$

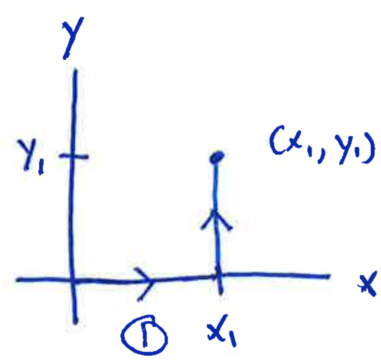
- In order for this vector field to be expressable as the gradient of a potential function,

$$\bar{E} = -\bar{\nabla}\phi$$

the necessary & sufficient condition is that the line integral of $\bar{E}$ around a closed loop is zero.

$$\oint \bar{E} \cdot d\bar{s} = 0$$

- Let's set up a square loop and compute the line integral along two different routes that terminate at the same x-y location.



First, lets go along the x-axis to x_1 then up to (x_1, y_1).

We'll call this path 1

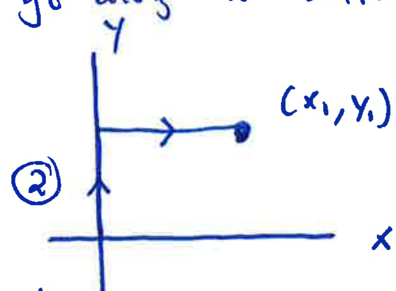
$$\underset{\text{along this path}}{\int \bar{E} \cdot d\bar{s}} = \int_0^{x_1} (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \cdot (dx\,\hat{x}) \quad (\text{with } y=0)$$

$$+ \int_0^{y_1} (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \cdot (dy\,\hat{y}) \quad (\text{with } x = x_1)$$

$$= \int_0^{x_1} (6x\cancel{y})\,dx + \int_0^{y_1} (\underbrace{3x^2}_{x=x_1} - 3y^2)\,dy \qquad y=0$$

$$\underset{\text{path ①}}{\int \bar{E} \cdot d\bar{s}} = 3x_1^2 y_1 - \frac{3y_1^3}{3} = 3x_1^2 y_1 - y_1^3$$

- Now lets go along a different path ...



we'll call this path ②

$$\int \bar{E} \cdot d\bar{s} = \int_0^{y_1} (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \cdot (dy\,\hat{y}) \quad (\text{with } x=0)$$

$$+ \int_0^{x_1} (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \cdot (dx\,\hat{x}) \quad (\text{with } y = y_1)$$

$$= \int_0^{y_1} (\underbrace{\cancel{3x^2}}_{x=0} - 3y^2)\,dy + \int_0^{x_1} (\underbrace{6xy}_{y=y_1})\,dx$$

$$\underset{\text{path ②}}{\int \bar{E} \cdot d\bar{s}} = -\frac{3y_1^3}{3} + \frac{6x_1^2 y_1}{2} = -y_1^3 + 3x_1^2 y_1$$

- Notice that $\int_{\text{path } ①} \bar{E} \cdot d\bar{s} = - \int_{\text{path } ②} \bar{E} \cdot d\bar{s}$

So the line integral of this vector field is path independent (at least for these two paths...)

- Reversing the direction of the second path just reverses the overall sign from ⊕ to ⊖, so we also notice that doing a complete circuit that begins and ends at the origin gives zero for the line integral.

$$\oint_{\text{closed loop.}} \bar{E} \cdot d\bar{s} = 0$$

which means I can define a potential function Φ such that $\bar{E} = -\bar{\nabla}\Phi$.

- In fact, I can just identify the potential function from either of our above integrations.

$$\boxed{\Phi(x,y) \equiv y^3 - 3x^2 y}$$

- Lets double check that this gives me back my vector field, $\bar{E}$.

$$E_x = -\frac{\partial \Phi}{\partial x} = 6xy \quad ✓$$

$$E_y = -\frac{\partial \Phi}{\partial y} = 3x^2 - 3y^2 \quad ✓$$

$$E_z = 0 \quad ✓$$