BB 2.7 (Modified wave equation)

Wave eqn: $\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial z^2}$ (for non-dispersive medium)

Modified: $\frac{\partial^2 \psi}{\partial t^2} = a \frac{\partial^2 \psi}{\partial z^2} - \omega_c^2 \psi$ (for dispersive media)

a) Assume solution $\psi = A e^{-j(\omega t - k z)}$

Does it work?

$$(j\omega)^2 \cancel{\psi} = a(-jk)^2 \cancel{\psi} - \omega_c^2 \cancel{\psi}$$

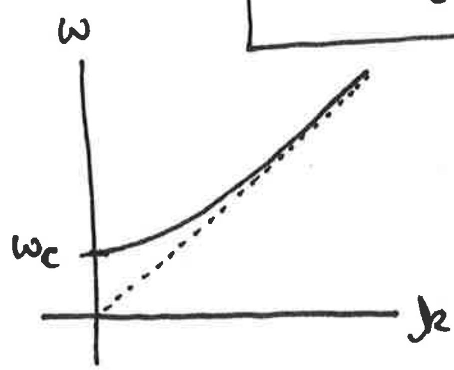
$$-\omega^2 = -ak^2 - \omega_c^2$$

$$\omega^2 = ak^2 + \omega_c^2$$

Our solution works, provided $\omega = \omega(k)$

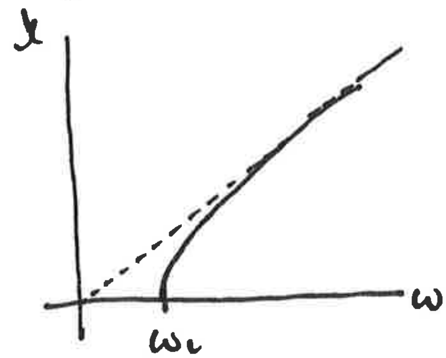
b) The dispersion relation is thus

$$\boxed{\omega = \omega_c \sqrt{1 + a\left(\frac{k}{\omega_c}\right)^2}}$$



This is linear as $k \to \infty$ but has a minimum value of $\omega = \omega_c$ @ $k = 0$ or $\lambda \to \infty$

we can also plot $k(\omega)$

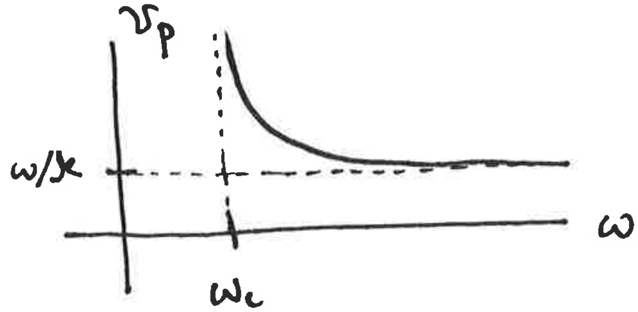


This implies that there is a cutoff frequency, ω_c, below which waves will not propagate.

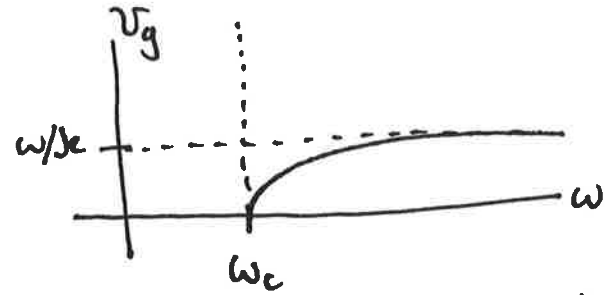
c) The <u>phase velocity</u> is defined as $v_p \equiv \frac{\omega}{k}$

and the <u>group velocity</u> is defined as $v_g \equiv \frac{\partial \omega}{\partial k}$

What do these look like?



The <u>phase velocity</u> is ω/k for large ω, and approaches <u>∞</u> as $\omega \rightarrow \omega_c$.



The <u>group velocity</u> is ω/k for large ω, and approaches <u>zero</u> as $\omega \rightarrow \omega_c$.

d) Since the group velocity tells us the rate at which energy is transported by the wave, this implies that there is <u>no energy transport</u> if $\omega < \omega_c$. So what happens? For $(\omega < \omega_c)$, the wave-number k becomes <u>purely</u> imaginary, which implies that the wave is attenuated/ <u>absorbed</u> for $\omega < \omega_c$ (instead of propagated).

- Let's look at what happens when $\underline{\omega < \omega_c}$ ....

$$k^2 = \frac{1}{a}\left(\omega^2 - \omega_c^2\right)$$

$$= \frac{-\omega_c^2}{a}\left(1 - \left(\frac{\omega}{\omega_c}\right)^2\right)$$

$$k = i \underbrace{\frac{\omega_c}{\sqrt{a}}\sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2}}_{K(\omega) = \text{ real number}}$$

$$k = i\,K(\omega)$$

- Our solution to the wave equation is

$$\psi = A e^{-j(\omega t - kz)}$$

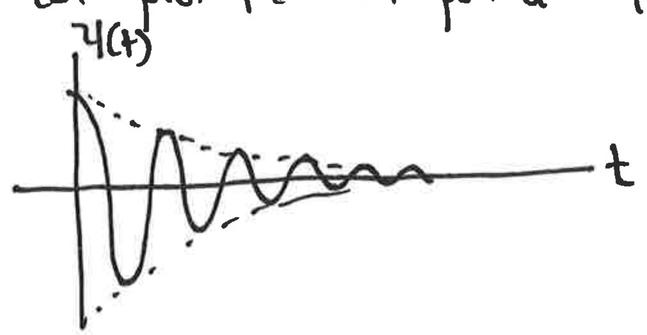
- If $\boxed{k = iK(\omega)}$ then this solution becomes.

$$\psi = A e^{-j\omega t} e^{+j(jK(\omega)z)}$$

$$\boxed{\psi = A \underbrace{e^{-j\omega t}}_{\text{oscillating}} \underbrace{e^{-K(\omega)z}}_{\text{exponentially decaying}}}$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

- We can plot the real part of ψ to get



So if $\omega < \omega_c$ we have a damped oscillation.