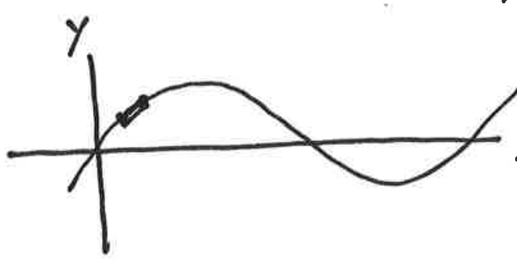
EX 2.2 (Characteristic impedance)

- Generally speaking, the impedance is a measure of the degree to which a thing is prevented from moving. This can be an electrical current, a mass of air, or a section of string.

(a) For a wave travelling down a string, the impedance of the string is defined as

$$Z_0 = \frac{\text{force on string in y-direction}}{\text{velocity in y-direction}}$$



Consider a wave traveling down a string under tension T. The mass per unit length is μ.

- From Chap. 2 of the book we found that the y-directed force on a small segment of string is

$$F_y = -T \frac{dy}{dx}$$

- If we take a wave $y(x,t) = y_0 e^{i(\omega t - kx)}$

then $\frac{dy}{dx} = -iky$

- The wave velocity is $v = \omega/k$ and the velocity of a section of string (in the y-direction) is $v_y = \frac{dy}{dt} = i\omega y$

- Going back to our definition of impedance and plugging all this in gives

$$Z_0 = \frac{-T\,dy/dx}{dy/dt} = \frac{-T(-iky)}{i\omega y}$$

$$Z_0 = Tk/\omega$$

- Using the fact that $v = \frac{\omega}{k}$ (generally)

and $v = \sqrt{T/\mu}$ (for a string)

we have

$$Z_0 = T\sqrt{\frac{\mu}{T}} = \sqrt{T\mu}$$

$$\boxed{Z_0 = \sqrt{\mu T}}$$

which is the impedance of a string!

(b) What about the impedance of electronic components, such as a capacitor? The <u>Impedance</u> is defined as the ratio of the electromotive force (voltage) to the speed of the electric charges (current)

$$Z_o = \frac{V}{I}$$

charge on capacitor

capacitance

For a capacitor, $Q = CV$ ← voltage across capacitor

$$\frac{dQ}{dt} = C\frac{dV}{dt}$$ ← taking time derivative

$$I = C\frac{dV}{dt}$$ ← current flowing into capacitor

Let's suppose $V = V_o e^{j\omega t}$ ← oscillating voltage

$$I = C\frac{d}{dt}(V_o e^{j\omega t}) = j\omega C V_o e^{j\omega t}$$

Now $$Z_o = \frac{V_o e^{j\omega t}}{j\omega C V_o e^{j\omega t}} = \frac{1}{j\omega C}$$

So $$\boxed{Z_o = \frac{1}{j\omega C}}$$ for a capacitor

c) For an inductor, the impedance is

$$Z_0 \equiv \frac{V_L}{I}$$

The voltage across an inductor is $V_L = L \frac{dI}{dt}$

Inductance

rate of change of current

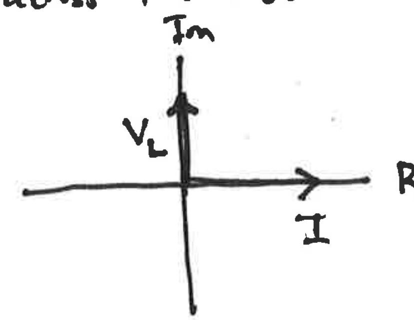
Suppose $I = I_0 e^{j\omega t}$

then $\frac{dI}{dt} = j\omega I$

so $Z_0 = \frac{j\omega L I}{I}$

$$\boxed{Z_0 = j\omega L}$$

<u>Aside</u>: Notice that this implies that the voltage across the inductor <u>leads</u> the current by 90°.



$V_L = Z_0 I_L$

$V_L = j\omega L\, I_0 e^{j\omega t}$

$V_L = \omega L I_0 e^{j(\omega t + \pi/2)}$

$$\left\{ \text{since } e^{j\pi/2} = \cancelto{0}{\cos\left(\tfrac{\pi}{2}\right)} + j\sin(\pi/2) \qquad e^{j\pi/2} = j \right\}$$

Thus $V_L = \omega L \overbrace{I_0 e}^{\text{current}}{}^{j\omega t + \pi/2}$ 90° phase shift