BB 1.4

a) Show $t - t_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x} \frac{dx}{\sqrt{E - V(x)}}$

where $x_1 = x(t_1)$

for conservative system $\frac{1}{2}mv^2 + V(x) = E$

$$\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 = E - V(x)$$

$$\frac{dx}{dt} = \sqrt{\frac{2\ E - V(x)}{m}}$$

$$\frac{dt}{dx} = \sqrt{\frac{m}{2}} \frac{1}{\sqrt{E - V(x)}}$$

$$\int_{t_1}^{t} dt = \sqrt{\frac{m}{2}} \int_{x_1}^{x} dx \frac{1}{\sqrt{E - V(x)}}$$

$$\boxed{t - t_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x} \frac{dx}{\sqrt{E - V(x)}}}$$

b) If $F(x) = -kx$

$$W_s = \int F(x')\,dx' = -V(x)$$

$$= \int_0^x -kx'\,dx' = -\frac{1}{2}kx^2$$

$$V(x) = \frac{1}{2}kx^2 = \boxed{\frac{1}{2}\omega_0^2 m x^2}$$

c)

$$t - t_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x} \frac{dx}{\sqrt{\frac{1}{2}kx_1^2 - \frac{1}{2}kx^2}}$$

$$= \sqrt{\frac{m}{k}} \int_{x_1}^{x} \frac{dx}{\sqrt{x_1^2 - x^2}}$$

$$= \frac{1}{\omega_0} \int_{x_1}^{x}$$

$$t - t_1 = \frac{1}{\omega_0} \int_{x_1}^{x} \frac{dx}{\sqrt{x_1^2 - x^2}}$$

$$= \frac{1}{\omega_0} \int_{x_1}^{x} \frac{dx}{x_1 \sqrt{1 - \left(\frac{x}{x_1}\right)^2}}$$

$$\left(\frac{x}{x_1}\right) \equiv \sin\theta \quad \boxed{, x = x_1 \sin\theta}$$

$$dx = x_1 \cos\theta \, d\theta$$

$$\int_{x_1}^{x} \frac{x_1 \cos\theta \, d\theta}{x_1 \sqrt{1 - \sin^2\theta}} = \int_{\theta_1}^{\theta} d\theta = \theta - \theta_1$$

$$t - t_1 = \frac{1}{\omega_0} \int_{x_1}^{x} \frac{dx}{\sqrt{x_1^2 - x^2}} = \frac{1}{\omega_0} (\theta - \theta_1)$$

$$\theta = \arcsin\left(\frac{x}{x_1}\right) \qquad \theta_1 = \arcsin(1) = \frac{\pi}{2}$$

$$t - t_1 = \frac{1}{\omega_0} \left[\arcsin\left(\frac{x}{x_1}\right) - \frac{\pi}{2} \right]$$

$$\omega_0 (t - t_1) + \frac{\pi}{2} = \arcsin\left(\frac{x}{x_1}\right)$$

$$x_1 \sin\left[\omega_0 (t - t_1) + \frac{\pi}{2}\right] = x$$

$$\boxed{x = x_1 \cos\left[\omega_0 (t - t_1)\right]}$$

We have shown that the previous is a solution to the S.H.O diff equation for $\alpha \approx 0$ $\quad \sin\alpha \approx \alpha \approx 0$

$$T \approx 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-0}} = 4\frac{\pi}{2}\sqrt{\frac{l}{g}}$$

$$\boxed{T \approx 2\pi\sqrt{\frac{l}{g}}}$$