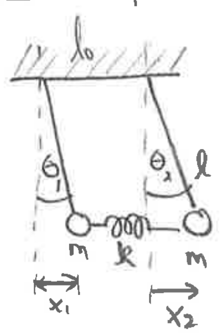
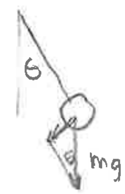
BB 1.14 Coupled oscillators



Newton's law: $(I\alpha = \text{Torque})$



$$ml^2\ddot{\theta} = mg\sin\theta\, l$$

$$\ddot{\theta} = \frac{g}{l}\theta$$

$$\Rightarrow \ddot{x} = \frac{g}{l}x \quad \longleftarrow \quad \theta \approx x$$

a) find diff equation governing coupled system:

$$m\ddot{x}_1 = -k(x_1-x_2) - \frac{mg}{l}x_1 \implies \ddot{x}_1 = -x_1\left(\frac{k}{m}+\frac{g}{l}\right) - x_2\left(\frac{-k}{m}\right)$$

$$m\ddot{x}_2 = -k(x_2-x_1) - \frac{mg}{l}x_2 \qquad \ddot{x}_2 = -x_1\left(\frac{-k}{m}\right) - x_2\left(\frac{k}{m}+\frac{g}{l}\right)$$

b) Obtain quadratic eqn for ω^2 and solve for ω.

$x_1 = A\cos(\omega t+\phi)$, $x_2 = B\cos(\omega t+\phi)$ $\longleftarrow$ guess solutions

$$\begin{aligned} -\omega^2 x_1 &= -a_{11}x_1 - a_{12}x_2 \\ -\omega^2 x_2 &= -a_{21}x_1 - a_{22}x_2 \end{aligned} \implies \begin{bmatrix} a_{11}-\omega^2 & a_{12} \\ a_{21} & a_{22}-\omega^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{\bar{A}}\bar{x} = 0$$

$$\det \bar{\bar{A}} = (a_{11}-\omega^2)(a_{22}-\omega^2) - a_{21}a_{12} = 0$$

$$a_{11}a_{22} + \omega^4 - a_{11}\omega^2 - a_{22}\omega^2 - a_{21}a_{12} = 0$$

$$\omega^4 + \omega^2(-a_{11}-a_{22}) + (a_{11}a_{22} - a_{21}a_{12}) = 0$$

$$\omega_1^2 = \frac{1}{2}(a_{11}+a_{22}) \pm \frac{1}{2}\sqrt{\underbrace{(a_{11}+a_{22})^2}_{a_{11}^2+a_{22}^2+2a_{11}a_{22}} - 4a_{11}a_{22} + 4a_{21}a_{12}}$$

$$\omega^2 = \frac{1}{2}(a_{11}+a_{22}) \pm \frac{1}{2}\sqrt{(a_{11}-a_{22})^2 + 4a_{21}a_{12}}$$

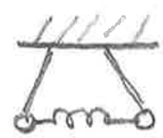
$a_{11} = a_{22}$

$$a_{11} = a_{22} = \frac{k}{m} + \frac{g}{l} \qquad a_{12} = a_{21} = -\frac{k}{m}$$

$$\omega^2 = \left(\frac{k}{m} + \frac{g}{l}\right) \pm \frac{1}{2}\sqrt{4\frac{k^2}{m^2}}$$

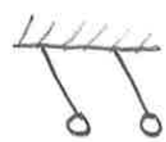
$$\omega^2 = \frac{k}{m} + \frac{g}{l} \pm \frac{k}{m}$$

$$\omega_1 = \sqrt{\frac{2k}{m} + \frac{g}{l}}$$



antisymmetric

$$\omega_2 = \sqrt{\frac{g}{l}}$$



symmetric

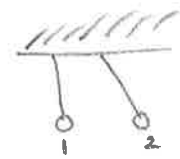
c) find ratio of amplitude of bobs

$$\frac{x_1}{x_2} = \frac{A}{B} = -\frac{a_{12}}{a_{11} - \omega^2} = \frac{k/m}{k/m + g/l - \left(\frac{k}{m} + \frac{g}{l} \pm \frac{k}{m}\right)}$$

$$\left.\frac{x_1}{x_2}\right|_S = \frac{k/m}{k/m} = 1$$

$$\left.\frac{x_1}{x_2}\right|_A = \frac{k/m}{-k/m} = -1$$

d) find solution w/ $\overbrace{\begin{matrix} x_2(0) = B, & x_1(0) = 0 \\ v_2(0) = 0, & v_1(0) = 0 \end{matrix}}^{\text{initial conditions}}$



(That is: we pull back a single pendulum & release.

i) $x_1 = A_S \cos(\omega_S t + \phi_S) + A_A \cos(\omega_A t + \phi_A)$ ← Position of pendulum 1 is a linear combination of symmetric & antisymmetric modes

ii) $x_2 = A_S \cos(\omega_S t + \phi_S) - A_A \cos(\omega_A t + \phi_A)$

$\cancel{\text{ii) } B = A_S \cos\phi_S - A_A \cos(\phi_A) = A_A (\cos\phi_S - \cos\phi_A)}$

$\cancel{\text{i) } 0 = A_S \cos\phi_S + A_A \cos\phi_A = A_A (\cos\phi_S + \cos\phi_A)}$

$\cancel{B = -2A_A \cos\phi_A} \Rightarrow A_A = \frac{-B}{2\cos\phi_A}$

$\cancel{\text{i) } 0 = -\omega_S A_S \sin\phi_S - \omega_A}$

$$
\begin{array}{rl} & \left[\begin{array}{l} 0 = A_S \cos\phi_S + A_A \cos\phi_A \\ B = A_S \cos\phi_S - A_A \cos\phi_A \end{array}\right] \\ - & \\ \hline \end{array}
$$
← enforce initial condition on positions of pendulae

$-B = 2A_A \cos\phi_A \qquad B = 2A_S \cos\phi_S \Rightarrow$

$$
\begin{array}{rl} & \left[\begin{array}{l} 0 = -\omega_S A_S \sin\phi_S - \omega_A A_A \sin\phi_A \\ 0 = -\omega_S A_S \sin\phi_S + \omega_A A_A \sin\phi_A \end{array}\right] \\ - & \\ \hline \end{array}
$$
← enforce initial conditions on velocity of pendulae

$0 = -2\omega_A A_A \sin\phi_A \Longrightarrow \phi_A = 0, \pi, 2\pi$

$\Rightarrow A_A = -\frac{B}{2}$ and $A_S = \frac{B}{2}$

$0 = -2\omega_S A_S \sin\phi_S \Rightarrow \phi_S = 0$

So $\phi_S = \phi_A = 0, \; A_S = B/2, \; A_A = -B/2$ ← solved for A_A, A_S, ϕ_S, ϕ_A

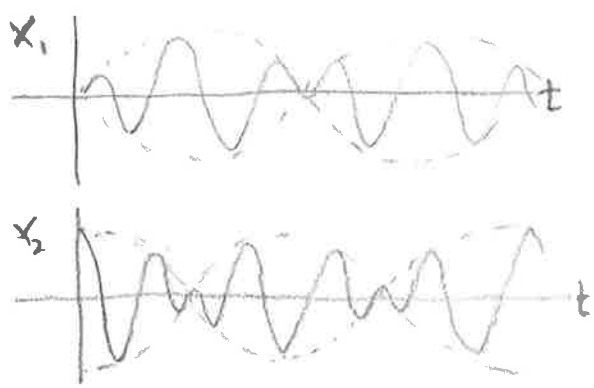
$$x_1 = \frac{B}{2}\left(\cos \omega_s t - \cos \omega_A t\right)$$

$$x_2 = \frac{B}{2}\left(\cos \omega_s t + \cos \omega_A t\right)$$

or

$$\boxed{\begin{aligned} x_1 &= -B \sin\left(\frac{\omega_s - \omega_A}{2} t\right) \sin\left(\frac{\omega_s + \omega_A}{2} t\right) \\ x_2 &= B \cos\left(\frac{\omega_s - \omega_A}{2} t\right) \cos\left(\frac{\omega_s + \omega_A}{2} t\right) \end{aligned}}$$

← Final solution



← Graphically

Notice that the pendulae are both oscillating, but the amplitudes of oscillation are 90° out of phase.