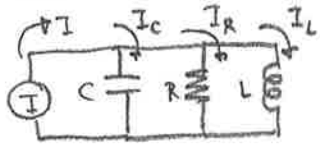
B.B 1.10 (4 pts)



$I = I_0 e^{i\omega t}$

a) differential equation for V_c?

$$I = I_c + I_R + I_L$$

$$Q_c = CV \Rightarrow I_c = \frac{dQ_c}{dt} = C\frac{dV}{dt}$$

$$I_R = \frac{V}{R}$$

$$V_L = L\frac{dI}{dt} \Rightarrow \frac{dI_L}{dt} = \frac{V_L}{L}$$

$$\frac{dI}{dt} = C\frac{d^2V}{dt^2} + \frac{1}{R}\frac{dV}{dt} + \frac{1}{L}V$$

$$\boxed{\frac{d^2V}{dt^2} + \beta\frac{dV}{dt} + \omega_0^2 V = \frac{i\omega}{C} I_0 e^{i\omega t}}$$

$\beta = \frac{1}{RC}$, $\omega_0^2 = \frac{1}{LC}$

b) $$V = \frac{(-\omega I_0/C)\cos(\omega t+\delta)}{\sqrt{(\omega^2-\omega_0^2)^2+(\beta\omega)^2}}$$

$$V(\omega=\omega_0) = V_{max} = \frac{(-\omega_0 I_0/C)\cos(\omega_0 t+\delta)}{\sqrt{\beta^2\omega_0^2}}$$

$$V_{max,0} = \frac{-I_0}{\beta C}$$

$$\frac{V_{max,0}}{2} = \frac{+I_0}{2\beta \cancel{C}} = \frac{(+\omega I_0/\cancel{C})}{\sqrt{(\omega^2-\omega_0^2)^2+(\beta\omega)^2}}$$

$$\sqrt{(\omega^2-\omega_0^2)^2+(\beta\omega)^2} = 2\beta\omega$$

$$(\omega^2-\omega_0^2)^2+(\beta\omega)^2 = 4(\beta\omega)^2$$

$$\sqrt{(\omega^2-\omega_0^2)^2} = \sqrt{3(\beta\omega)^2}$$

$$\omega^2-\omega_0^2-\sqrt{3}\beta\omega = 0$$

$$\omega = \frac{\sqrt{3}\beta}{2} \pm \frac{1}{2}\sqrt{3\beta^2+4\omega_0^2}$$

$$\boxed{\omega = \frac{\sqrt{3}}{2}\beta \pm \frac{\sqrt{3}}{2}\beta\sqrt{1+\frac{4}{3}\left(\frac{\omega_0}{\beta}\right)^2}}$$

If $R = 1\Omega$, $L = 10^{-4}H$, $C = 10^{-8}F$ then $\omega_0^2 = \frac{1}{LC} = 10^{12}$

$\omega_0 = 10^6$

$\beta = \frac{1}{RC} = \frac{1}{10^9} = 10^{+9}$

$(\omega_0/\beta) = 10^{-3} << 1$

$$\sqrt{1+\frac{4}{3}\left(\frac{\omega_0}{\beta}\right)^2} \approx 1+\frac{1}{2}\,\frac{4}{3}\left(\frac{\omega_0}{\beta}\right)^2 - \frac{1}{8}\,\frac{16}{9}\left(\frac{\omega_0}{\beta}\right)^4$$

$$(\omega^2-\omega_0^2)^2+(\beta\omega)^2 = 4(\beta\omega)^2$$

$$\omega^4+\omega_0^4-2\omega^2\omega_0^2-3\beta^2\omega^2=0$$

$$\omega^4+\omega^2(-2\omega_0^2-3\beta^2)+\omega_0^4=0$$

$$\omega^2 = \frac{1}{2}(2\omega_0^2+3\beta^2) \pm \frac{1}{2}\sqrt{\cancel{4\omega_0^4}+9\beta^4+12\omega_0^2\beta^2-\cancel{4\omega_0^4}}$$

$$\omega^2 = \omega_0^2+\frac{3\beta^2}{2} \pm \frac{3\beta^2}{2}\sqrt{1+\frac{4}{3}\left(\frac{\omega_0}{\beta}\right)^2}$$

$$\approx \omega_0^2+\frac{3}{2}\beta^2 \pm \frac{3\beta^2}{2}\left(1+\frac{2}{3}\left(\frac{\omega_0}{\beta}\right)^2-\frac{2}{9}\left(\frac{\omega_0}{\beta}\right)^4\right)$$

$$\omega_1^2 = \omega_0^2+3\beta^2+\frac{2}{3}\omega_0^2 - \frac{1}{3}\frac{\omega_0^4}{\beta^2} \approx \frac{5}{3}\omega_0^2+3\beta^2-\frac{1}{3}\frac{\omega_0^4}{\beta^2}$$

$$\omega_2^2 = \omega_0^2-\frac{2}{3}\omega_0^2+\frac{1}{3}\frac{\omega_0^4}{\beta^2} \approx \frac{1}{3}\omega_0^2+\frac{1}{3}\frac{\omega_0^4}{\beta^2}$$

$$\omega_1^2 = \frac{5}{3}\omega_0^2\left(1-\frac{1}{5}\left(\frac{\omega_0}{\beta}\right)^2\right)+3\beta^2$$

$$\omega_2^2 = \frac{1}{3}\omega_0^2\left(1+\left(\frac{\omega_0}{\beta}\right)^2\right)$$

$$\boxed{\begin{aligned}\omega_1 &= \sqrt{\tfrac{5}{3}}\,\omega_0\\ \omega_2 &= \sqrt{\tfrac{1}{3}}\,\omega_0\end{aligned}}$$

c) $Q \approx \omega_0/\beta = \frac{1}{\sqrt{LC}} RC = R\sqrt{\frac{C}{L}}$

or

$\Delta\omega = \omega_2 - \omega_1 = \sqrt{\frac{5}{3}}\,\omega_0 - \sqrt{\frac{1}{3}}\,\omega_0$

$= \frac{\omega_0}{\sqrt{3}}(\sqrt{5}-1)$

$\Delta\omega = 0.71\,\omega_0$

$Q = \frac{\omega_0}{0.71\,\omega_0}$

$\boxed{Q = 1.4}$